

The Fractional Fourier Transform in Information Optics

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Abstract

The fractional Fourier transform is an important tool for both signal processing and optical communities. This paper presents a tutorial which includes the major related aspects of this transformation.

1. Introduction

The fractional Fourier transform (FRT) operation was shown to be useful for various spatial filtering and signal processing applications [1-8]. The FRT is a private case of the ABCD matrix. When the ABCD matrix accepts the form of:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

the ABCD transform becomes the FRT.

In this transform the amount of shift variance may be controlled by choosing the proper fractional order p for the transformation while ϕ equals to $p\pi/2$. When the fractional order is one, the FRT becomes the conventional Fourier transform that is totally shift invariant. For fractional order of zero the FRT gives the input function, i.e. totally shift variant. For any other fractional orders in between, the transform has a partial amount of shift variance.

2. FRT- Definition

There are two common interpretations for the FRT. Both definitions were proven to be identical as shown in Ref. [7].

2.1 Definition based on propagation in graded index media

The first FRT definition [9-11] is based on the field propagating along a quadratic graded index (GRIN) medium having a length proportional to p (p being the FRT order). The eigen-modes of quadratic GRIN media are the Hermite-Gaussian (HG) functions, which form an orthogonal and complete basis set. The m th member of this set is expressed as:

$$\Psi_m(x) = H_m\left(\frac{\sqrt{2}x}{\omega}\right) \exp\left(-\frac{x^2}{\omega^2}\right)$$

where H_m is a Hermite polynomial of order m and ω is a constant associated with the GRIN medium parameters. An extension to two lateral coordinates x and y is straightforward, with $\psi_m(x)\psi_n(y)$ as elementary functions.

The propagation constant for each HG mode is given by:

$$\beta_m = k\sqrt{1 - \frac{2}{k}\sqrt{\frac{n_2}{n_1}}\left(m + \frac{1}{2}\right)} \approx k - \sqrt{\frac{n_2}{n_1}}\left(m + \frac{1}{2}\right)$$

with $k=2\pi/\lambda$ and λ is the wavelength. The HG set is used to decompose any arbitrary distribution $u(x)$

$$u(x) = \sum_m A_m \Psi_m(x)$$

where the coefficient A_m of each mode $\psi_m(x)$ is given by:

$$A_m = \int_{-\infty}^{\infty} u(x)\Psi_m(x) / h_m dx$$

with

$$h_m = 2^m m! \sqrt{\pi\omega} / \sqrt{2}$$

Using the above decomposition, the FRT of order p is defined as:

$$\mathfrak{T}^p[u](x) = \sum_m A_m \Psi_m(x) \exp(i\beta_m p L)$$

$$L = \frac{\pi}{2} \sqrt{\frac{n_1}{n_2}}$$

L is the GRIN length that results in the conventional Fourier transform. It was shown [10] that this definition agrees well with the classical Fourier transform definition when $p=1$.

2.2 Definition based on Wigner distribution function

A complete signal description, displaying space and frequency information simultaneously, can be achieved by the space-frequency Wigner distribution function (WDF) [12].

In Ref. [13] the fractional Fourier transform operation is defined by following the signal $u(x)$ while its WDF is rotated by an angle $\phi = p\pi/2$. Note that the WDF of a 1-D function is a 2-D function and the rotation interpretation is easily displayed.

In Ref. [13], the same rotation strategy was generalized to 2-D signals, i.e. images, whose WDFs are 4-D distributions. The WDF of a function can be rotated with bulk optics [13].

This optical setup represents in the WDF space three shearing operations: x, v -shearing or v, x -shearing. Where v is the spectral coordinate and x is the spatial one. The x -shearing is performed by free-space propagation, then a lens performs v -shearing, then again x -shearing is performed by free-space propagation. By analyzing the optical configuration of Ref. [13], Lohmann obtained:

$$u_p(x) = \mathfrak{F}^p[u(x_0)] = C_1 \int u(x_0) \exp\left(i\pi \frac{x_0^2 + x^2}{\lambda f_1 \tan \phi}\right) \exp\left(-i2\pi \frac{xx_0}{\lambda f_1 \sin \phi}\right) dx_0$$

with

$$C_1 = \frac{\exp\left[-i\left(\frac{\pi \operatorname{sgn}(\sin \phi)}{4}\right) - \frac{\phi}{2}\right]}{\sqrt{|\lambda f_1 \sin \phi|}}$$

This last equation defines the FRT for 1-D functions. Generalization for 2-D functions is straightforward. Note that λf_1 is also coined the scaling factor.

The two interpretations of the FRT operation have been united into one formulation through a transformation kernel, as illustrated in Ref. [8]:

$$u_p(x) = \mathfrak{F}^p[u(x')](x) = \int B_p(x, x') u(x') dx'$$

where $B_p(x, x')$ is the kernel of the transformation and p is the fractional order. The kernel has two optical interpretations, one as a propagation through GRIN medium [10]

$$B_p(x, x') = \sqrt{2} \exp\left[-\frac{x^2 + x'^2}{\omega}\right] \sum_{n=0}^{\infty} \frac{i^{-pn}}{2^n n!} H_n\left(\frac{\sqrt{2}x}{\omega}\right) H_n\left(\frac{\sqrt{2}x'}{\omega}\right)$$

and the second as a rotation operation applied over the Wigner plane [13]

$$B_p(x, x') = C_1 \exp\left[i\pi \left(\frac{x^2 + x'^2}{\lambda f_1 \tan \phi}\right) - 2\pi i \left(\frac{xx'}{\lambda f_1 \sin \phi}\right)\right]$$

Note that ω is the coefficient that connects the two interpretations:

$$\omega = \sqrt{\frac{\lambda f_1}{\pi}}$$

3. Properties of the FRT

- *Linearity:*

The FRT of a linear combination of two input functions u_1 and u_2 behaves according to the definition of linear systems. c_1 and c_2 are constants.

$$\mathfrak{F}^p [c_1 u_1(x) + c_2 u_2(x)] = c_1 \mathfrak{F}^p [u_1(x)] + c_2 \mathfrak{F}^p [u_2(x)]$$

- *Continuity:*

Two FRTs with different orders p_1 and p_2 yield the following theorem:

$$\mathfrak{F}^{(c_1 p_1 + c_2 p_2)} [u(x)] = \mathfrak{F}^{c_1 p_1} [\mathfrak{F}^{c_2 p_2} (u(x))] = \mathfrak{F}^{c_2 p_2} [\mathfrak{F}^{c_1 p_1} (u(x))]$$

- *Parseval's theorem:*

$$\int |u_0(x_0)|^2 dx_0 = \int |u_p(x_p)|^2 dx_p$$

- *Shift theorem:*

If the input object is shifted by the amount of a , then its FRT yields:

$$u_p(x; a) = \mathfrak{F}^p [u_0(x_0 + a)] = \exp[i\pi a \sin \phi (2x + a \cos \phi)] u_p(x + a \cos \phi)$$

- *Scaling theorem:*

If the input object is scaled by the factor of a , then its FRT yields:

$$u_p(x; a) = \mathfrak{F}^p [u_0(ax_0)] = \frac{\psi}{a} u_{\hat{p}} \left(\frac{x}{\sqrt{\sin^2 \phi \left(a^2 - \frac{1}{a^2} \right) + \frac{1}{a^2}}} \right)$$

where

$$\hat{p} = \frac{2}{\pi} \tan^{-1} \left[a^2 \tan \left(\frac{\pi p}{2} \right) \right]$$

$$\psi = \exp \left[\frac{i\pi x^2 (a^4 - 1)}{a^4 \tan \phi + \frac{1}{\tan \phi}} \right]$$

4. References

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