ALGEBRAIC INVARIANTS AND THEIR USE IN AUTOMATIC IMAGE RECOGNITION

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ABSTRACT

Automatic recognition of objects independent of size, orientation, position in the field of view, and color is a difficult and important problem in computer vision, image analysis, and automatic target recognition fields. In this paper we explore the theory of invariant algebra to develop solutions for this problem. Algebraic invariants of binary and ternary quantics are used to obtain features that remain unchanged when the object undergoes linear geometrical and spectral transformations. Empirical examples of the use of this approach on real and synthetic data are provided.

Keywords- Automatic Target Recognition, invariant algebra, pattern recognition, signal processing.

1. INTRODUCTION

Automatic recognition of objects independent of size, orientation, position in the field of view, and color is a difficult and important problem in computer vision, image analysis, and automatic target recognition fields. A direct approach to this problem is by use of a large library of target signatures at all potential positions, viewing angles, spectral bands and contrast conditions that can lead to a combinatorial explosion of models to be considered. Another approach is by development of composite template filters by means of which potential viewing instances of a target under differing size, orientation, spectral and contrast variations are used to create a single composite template filter that is then used for detection and classification of that target.

In this paper we consider a third approach by exploring the theory of invariant algebra to develop solutions for this problem. Algebraic invariants of binary and ternary quantics are used to develop features that remain unchanged when the object undergoes linear geometrical and spectral transformations. Invariant algebra is a mathematical discipline that arises in relation with a number of problems in algebra and geometry. Extractions of algebraic expressions that remain unchanged under changes of coordinate systems are part of this discipline. Lagrange seems to be among the first mathematicians who first studied invariants.

In the following section we provide a background on invariant algebra, develop the concept of object representation in terms of a probability density function and its statistical moments and present invariants of binary and ternary quantics. Section 3 provides an application of invariants of binary quantics in geometrical and spectral invariant object representation. Section 4 provides an application of ternary quantics invariants in 3D Ladar target classification. Finally, in Section 5, we provide a summary of the main finding of this paper.

2. THEORY OF INVARIANT ALAGEBRA

Studying the intrinsic properties of polynomials, that remain undisturbed under changes of variables, forms the domain of this theory. The study and derivation of the algebraic invariants has a long history, which goes back to Lagrange and Boole. However, its development as an independent discipline is due to the work of Cayley and Sylvester in the 19^{th} Century [1, 2, 3].

Consider a homogenous nth order polynomial of m variables. In the parlance of invariant algebra this polynomial is referred to as an m-ary quantic of order n (or m-ary n-ic). The goal pursued under this theory is the derivation of those algebraic expressions of the coefficients of this quantic that remain invariant when the m variables undergo a linear transformation. The coefficients of the transformation act as a multiplying factor. When this factor is eliminated the invariants are referred to as absolute invariants

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As an example consider a ternary quantic of order m:

$$f(x_1,x_2,x_3) = \sum_{\substack{p,q,r=0 \\ p+q+r=m}}^{m} \frac{m!}{p!q!r!} a_{pqr} x_1^p x_2^q x_3^r \tag{1}$$

A homogenous polynomial I(a) of coefficients is called an invariant of an algebraic form f if , after transforming its set of variables from x to x' , and constructing a corresponding polynomial I(a') of the new coefficients the following holds true:

$$I(a)=\Lambda I(a') \tag{2}$$

Λ is independent of the f(x) and depends only on the transformation. For homogenous polynomials considered here $\Lambda=\Delta^{\omega}$, where Δ is the determinant of the transformation and ω is called the weight of the invariant. The invariant is called absolute when $\omega=0$.

Any object in a multi-dimensional coordinate system (x_1, x_2, $x_{3\ldots}$) can be represented in terms of a probability density function $\rho(x_1, x_2, x_{3\ldots})$ by proper normalization. Moreover it is well known that any probability density function (PDF) can be uniquely defined in terms of its infinite statistical moments[3].

Multi-dimensional moment of order p+q+r+… of a density $\rho(x_1, x_2, x_{3\ldots})$ is defined as the Riemann integral as:

$$m_{pqr\ldots} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots x_1^p x_2^q x_3^r \ldots \rho(x_1,x_2,x_3\ldots) dx_1 dx_2 dx\ldots \tag{3}$$

The sequence of $\{m_{pqr\ldots}\}$ determines uniquely $r(x_1, x_2, x_{3,\ldots})$.

Using the definition of moment generating function of multi-dimensional moments and expanding it into a power series one has the following:

$$M(u_1,u_2,u_3,\ldots) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots \sum_{p=0}^{\infty} \tfrac{1}{p!}(u_1x_1 + u_2x_2 + u_3x_3\ldots)^p \rho(x_1,x_2,x_3,\ldots) dx_1 dx_2 dx_3\ldots \tag{4}$$

This by a few algebraic manipulations is reduced to an n-ary quantic of order m similar to (1).

Fundamental Theorem of Moment Invariants-

If a m-ary p-ic (a homogeneous polynomial of order p in m variables) has an invariant:

$$f(a_{p\ldots0},\ldots a_{0\ldots p}) = \Delta^{\omega} f(a_{p\ldots0},\ldots a_{0\ldots p}) \tag{5}$$

Then the moment of order p has an algebraic invariant:

$$f(\mu_{p\ldots0},\ldots\mu_{0\ldots P}) = |j|\Delta^{\omega} f(\mu_{p\ldots0},\ldots\mu_{0\ldots p}) \tag{6}$$

Where J is the Jacobian of the transformation.

For the case of Binary quantic the following invariants can be derived[6]:

$$
\begin{aligned}
&\phi_{1+} = \eta_{20} + \eta_{02} \\
&\phi_2 = (\eta_{20} - \eta_{02})^2 + 4\eta_{11}^2 \\
&\phi_3 = (\eta_{03} - 3\eta_{12})^2 + (3\eta_{21} + \eta_{03})^2 \\
&\phi_4 = (\eta_{30} + \eta_{12})^2 + (\eta_{21} + \eta_{03})^2 \\
&\phi_5 = (\eta_{30} - 3\eta_{12})(\eta_{03} + \eta_{30})[(\eta_{30} + \eta_{12})^2 - 3(\eta_{21} + \eta_{03})^2] \\
&+ (3\eta_{21} - \eta_{03})(\eta_{21} + \eta_{03})[3(\eta_{30} + \eta_{12})^2 - (\eta_{21} + \eta_{03})^2] \\
&\phi_6 = (\eta_{20} - \eta_{02})[(\eta_{30} + \eta_{12})^2 - (\eta_{21} + \eta_{03})^2] \\
&+ 4\eta_{11}(\eta_{30} + \eta_{12})(\eta_{21} + \eta_{03})
\end{aligned} \tag{7}
$$

A seventh invariant can be added that will change sign under "improper" rotation.

$$
\begin{aligned}
&\phi_7 = (3\eta_{12} - \eta_{30})(\eta_{30} + \eta_{12})[(\eta_{30} + \eta_{12})^2 - 3(\eta_{21} + \eta_{03})^2] \\
&+ (3\eta_{21} - \eta_{03})(\eta_{21} + \eta_{03})[3(\eta_{30} + \eta_{12})^2 - (\eta_{21} + \eta_{03})^2]
\end{aligned} \tag{8}
$$

ϕ s are related to μ by the following normalization factor that will make the central moments invariant under size change:

$$
\eta_{pq} = \frac{\mu_{pq}}{\mu_{00}^{\frac{(p+q)}{2}+1}} \tag{9}
$$

For the case of ternary quadratics the following invariants are derived[2]:

$$
\begin{aligned}
&J_{1\mu} = \mu_{200} + \mu_{020} + \mu_{002} \\
&J_{2\mu} = \mu_{020}\mu_{002} - \mu_{001}^2 + \mu_{200}\mu_{002} - \mu_{101}^2 + \\
&\mu_{200}\mu_{020} - \mu_{110}^2 \\
&\Delta_{2\mu} = \det\begin{pmatrix} \mu_{200} & \mu_{110} & \mu_{101} \\ \mu_{110} & \mu_{020} & \mu_{011} \\ \mu_{101} & \mu_{011} & \mu_{002} \end{pmatrix}
\end{aligned} \tag{10}
$$

Where μ_{pqr} denote the centralized moments. The following absolute invariants are then obtained by simple algebraic manipulations:

$$
I_3 = \frac{J_1 J_2}{\Delta_2} \qquad I_1 = \frac{J_1^2}{J_2} \quad \text{or} \quad I_1 = \frac{J_1^2}{J_2} \qquad I_2 = \frac{\Delta_2}{J_1^3} \tag{11}
$$

3. APPLICATIONS OF INVARIANTS OF BINARY QUANTICS

Two-dimensional Geometrical Invariancy- The following Figure show an airborne view of a military truck viewed in three different field of view geometries. Fig 1(a) shows the first view. Fig 1 (b) shows the same scene when the field of view is rotated 90 degrees. Fig 1 (c) show the same scene again when the field of view is rotated 180 degrees. The corresponding invariant expressions for these three scenes are computed using the relations (7) and (8). These values are tabulated in Table 1. As can be seen from this Table the computed invariants remain mostly unchanged when the field of the view of the scene is changed by 90 and 180 degrees.

Original Scene (a) | Scene 90 degrees Rotated (b) | Scene 180 degrees Rotated (c)

Fig 1. A scene under 3 different rotation states.

Table 1. Invariant values for a scene under three different geometries

Invariants	Original Scene	Scene Rotated by 90 Degrees	Scene Rotated by 180 Degrees
Φ_1	5344.64	5344.64	5344.64
Φ_2	1112.63	1112.63	1112.63
Φ_3	7.63	7.63	7.69
Φ_4	0.72	0.72	0.72
Φ_5	-1.69	-1.69	-1.69
Φ_6	-16.90	-16.90	-16.90
Φ_7	-1.21	1.21	-1.21

Joint Geometrical and Material Invariancy- from the Planck's Law[7], (which in its common form does not have the spectral emissivity included in it), one has the following relationship between the emissivity, temperature, wavelength, and the spectral radiant emittance W_λ.

$$W_\lambda = \frac{2\pi hc^2}{\lambda^5} \frac{\varepsilon_\lambda}{e^{\frac{ch}{\lambda KT}} - 1} \tag{12}$$

where W_λ, the spectral radiant emittance is in Watts cm^{-2} micrometer $^{-1}$. T is the absolute temperature in degree Kelvin, ε_λ is spectral emissivity, h is the Planck's constant = $(6.6256 \pm 0.0005) \times 10^{-34}$ Watts sec^2, λ is the wavelength in centimeter, K is the Boltzman constant=$(1.38054 \pm 0.00018) \times 10^{-23}$ Watts sec./degree Kelvin and c is the speed of light in cm /sec. In (1) only emissivity is material dependent. Emissivity, defined as the ratio of the radiant emittance of the illuminating source to the radiant emittance of the black body, is dimensionless, and has a value between 0 and 1.

The scene radiation is obtained by integrating (12) over different wavelength bands. From a wide range of wavelength bands, different spectral images corresponding to the same scene are obtained.

When the effects of the radiation reflectance are negligible or ignored and we assume that the scene is in thermal equilibrium, the radiation varies only with ε_λ in a particular scene. The variations of ε_λ with frequency for different materials and paints are well documented.

The output of a focal plane array (FPA), in general, is a linear function of the incidence photons emanating from the scene:

$$N_{ij} = K_{ij} \int_{\lambda_1}^{\lambda_2} \zeta_{ij}(\lambda) \{\varepsilon_\lambda(i,j) W_\lambda(T_{ij}) + [1-\varepsilon_\lambda(i,j)] W_\lambda(T_b)\} d\lambda + N_{ij}^d \tag{13}$$

where N_{ij} is the total number of accumulated electrons at pixel ij, K_{ij} is a coefficient that is dependent on the active pixel area, optical transmission, frame time, and pixel angular displacement from the optical axis, and the f-number of the optics. The quantum efficiency of the ij pixel is denoted by $\zeta_{ij}(\lambda)$, and background radiant reflectance is shown by $W_{\lambda}(T_b)$. T_b is the background temperature in degrees Kelvin, and λ_1 and λ_2 define the spectral band of the sensor. Finally, the dark charge for the pixel ij is denoted by N^{d}_{ij}. The $\varepsilon_{\lambda}(i,j)$ indicates the spectral emissivity at the pixel location ij.

At each pixel location ij, consider ε(i,j), as an n-dimensional vector (n being the number of wavelengths used). Then, consider the probability of it being from a material π_K k being the number of different materials in the scene, be denoted as $p(\varepsilon|\pi_k)$ and the probability of material occurrence π_K as $p(\pi_K)$. Then according to the Bayes decision rule, one has to select the following:

$$\max_k \{p(\pi_k|\varepsilon)\} = \max_k \{\frac{p(\varepsilon|\pi_k)p(\pi_k)}{p(\varepsilon)}\} \tag{14}$$

$p(\varepsilon|\pi_k)$ is assumed to be known for each frequency k at a range of temperatures of interest. This assumption is not restrictive since for different material (and paints) the emissivity as function of frequency and temperature has been documented. The p(ε) is obtained from the following:

$$p(\varepsilon) = \sum_{\pi_k} p(\varepsilon|\pi_k)p(\pi_k) \tag{15}$$

Once for each pixel location, a material label has been chosen a new image is formed. This image is formed by replacing the value of each pixel with its most likely emissivity label (iron=1, water=2, etc.). Denoting each pixel as $\pi_k(i,j)$, the information content of the image varies by the frequency of occurrence of the emissivity label in the image.

In the above case the invariant expressions (7) and (8) depend on the material that the targets and scene are made of. Moreover they are k-ary quantics (homogeneous polynomials of k variables). Consequently, for any linear transformation in π_k (changes in material mixtures) there exist a set of algebraic expressions that will remain unchanged. These second order invariants will be invariant under scene rotation, scale, translation and material mixture transformations. The expressions $\phi_1(\pi_k)$ to $\phi_7(\pi_k)$ are polynomials of order 1 to 4 in terms of π_k for various k. Each of different k values indicates a different material. Any linear transformation of π_k indicates a change of material mixture in the scene such as changing the paint on a target, or having the objects on a dry land versus wetland, or for a target being on a grass verses being on a concrete background.

Example- Consider an object whose image is represented analytically as by the following function:

$$f(x,y,p,q) = (p+q)e^{-x-y} \tag{16}$$

Where x and y represent the axes for the object's geometry and p and q represent the axis for each pixel's material mixtures. In the following we will refer to the objects change of orientation, scale and position as its geometrical transformation to distinguish this type of change from those associated with the object's surface material.

Changes in the object's orientation, scale and positions in the field of view will lead to changes in the object image, and consequently in its representation, as has been expressed in the equation (16). Similarly, any change in the surface material of the object (for example a differing surface paint) also causes a change in the object's image representation as expressed in the equation (16).

In the following, we will derive the geometrical invariants for this object and from them will extract the material invariants associated for each pixel on the object. We will then change the material mixture arbitrarily and show the invariancy of these expressions to the joint geometrical and material transformations.

Deriving the geometrical invariants for the object, whose image is represented by the equation (14), one can obtain the following relationships by using Equations (7), and (8):

$$\text{GeoInvariant 1}=(2(1.15(p+q)-0.57(p+q)^2+0.09(p+q)^3))/(1.0.33(p+q)) \tag{17}$$

$$\text{GeoInvariant 2}=(4(p+q)^2(0.84-0.57(p+q)+(0.09(p+q)^2)^2)/(1.03(p+q))^2 \tag{18}$$

$$\text{GeoInvariant 3}=(0.55(-1.10+p+q))(-1.10+p+q))(p+q)^2\ (17.60-7.75(p+q)+(p+q)^2))/(5.07+(p+q)^{1.5})^2 \tag{19}$$

$$\text{GeoInvariant 4}=(0.2788(p+q)^2(15.9731-7.9674(p+q)+(p+q)^2)(18.1287-6.2169(p+q)+(p+q)^2)(9.8189-3.5321(p+q)+(p+q)^2))/(5.0739+(p+q)^{1.5})^2 \tag{20}$$

$$\text{GeoInvariant 5}=(0.38(-3.8+p+q)(-1.10+p+q)(p+q)^4(17.60-7.75(p+q)+(p+q)^2)(15.42-5.60(p+q)+(p+q)^2)(13.27-5.02(p+q)+(p+q)^2)(11.40-4.36(p+q)+(p+q)^2))/(5.07+(p+q)^{1.5})^4 \tag{21}$$

$$\text{GeoInvariant 6}=(-0.16(-3.83+p+q)(-2.95+p+q)(-2.95+p+q)(p+q)^4(14.72-7.67(p+q)+(p+q)^2)(13.27-5.02(p+q)+(p+q)^2)(13.27-5.02(p+q)+(p+q)^2)(13.27-5.02(p+q)+(p+q)^2))/((2.95+(p+q))(5.07+(p+q)^{1.5})^3) \tag{22}$$

$$\text{GeoInvariant 7}=(0.69(-4.11+p+q)(-3.95+p+q)(-3.83+p+q)(-1.10+p+q)(p+q)^4(17.60-7.75(P+q)+(p+q)^2)(13.277-5.02(p+q)+(p+q)^2)(14.21-4.74(p+q)+(p+q)^2))/(5.07+(p+q)^{1.5})^4 \tag{23}$$

We next change the object's materials via a rotation of its basis vectors p and q, by 180 degrees. The resulting material invariant values, tabulated in Table 2, illustrate clearly that the material invariants remain unchanged to a high degree of precision. It should be noted that the material invariants, by virtue of being derived from the geometrical invariants, are unchanged also under object's rotation, translation and scale change in the sensor's field of view.

4. INVARIANT OF TERNARY QUANTICS

A set of Ladar targets composed of a total of 35 tactical military targets encompassing tanks, trucks, APCs, self-propelled guns from the US and other countries were used in the experiment. In this study mostly one resolution representing sensor-to-scene distance of 100 meters was considered. Figures 2 shows sample Ladar images of some of these targets. The ground plane template for each of the image scenes is a 20m x 20m flat surface placed on the XY plane. The maximum height was 10m. To explore the effects of noise a Gaussian probability density function of zero mean with varying variances was added to the coordinates of each point in the point clouds. A standard deviation value of 1 corresponds to a distance of 1 meter.

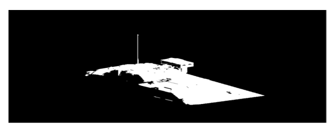
Table 2- Joint Geometrical and Material Invariants

	Inv1	Inv2	Inv3	Inv4	Inv5	Inv6	Inv7
GeoInv 1	11571.435	20314148	188349.36	27209.686	-1.946E+09	-122551302	89242710
	11571.435	20314148	188349.36	27209.686	-1.946E+09	-122551302	89242710
GeoInv 2	33861.86	88715529	250759.07	2678.8649	-68932078	-25108715	8707525.9
	33861.86	88715529	250759.07	2678.8649	-68932078	-25108715	8707525.9
GeoInv 3	11341.667	19448072	184326.73	27161.357	-1.921E+09	-119747842	54648854
	11341.667	19448072	184326.73	27161.357	-1.921E+09	-119747842	54648854
GeoInv 4	17021.829	17355576	615048.93	172176.8	-5.59E+10	-715782571	3.311E+09
	17021.829	17355576	615048.93	172176.8	-5.59E+10	-715782571	3.311E+09
GeoInv5	16648.762	17988388	580883.58	156267.49	-4.699E+10	-661440009	2.845E+09
	16648.762	17988388	580883.58	156267.49	-4.699E+10	-661440009	2.845E+09
GeoInv 6	19571.193	16468142	783278.67	233664.72	-9.99E+10	-947133124	-2.735E+09
	19571.193	16468142	783278.67	233664.72	-9.99E+10	-947133124	-2.735E+09
GeoInv 7	28642.364	16119054	824001.26	247466.68	-1.115E+11	-989943688	5.617E+09
	28642.364	16119054	824001.26	247466.68	-1.115E+11	-989943688	5.617E+09

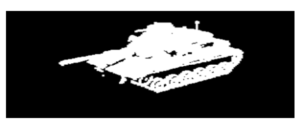
M110 A2 Self-Propelled Gun

BMP

M1 A2 Abrams



M2 A2 Bradley



M60 A3

Fig.2. Sample of Targets used in the Experiments

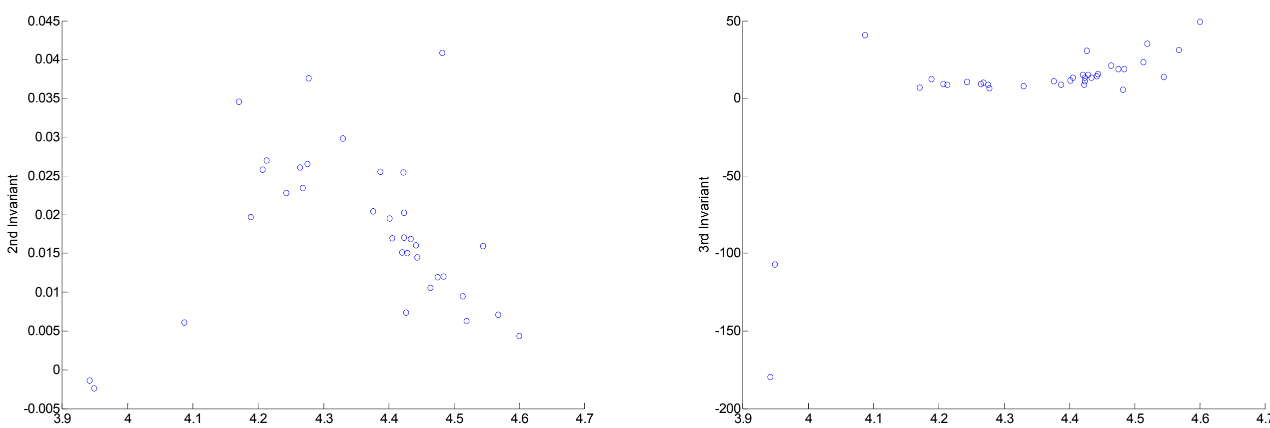


Fig 4. Distribution of the 2nd and 1st and the 3rd and 1st absolute invariants.

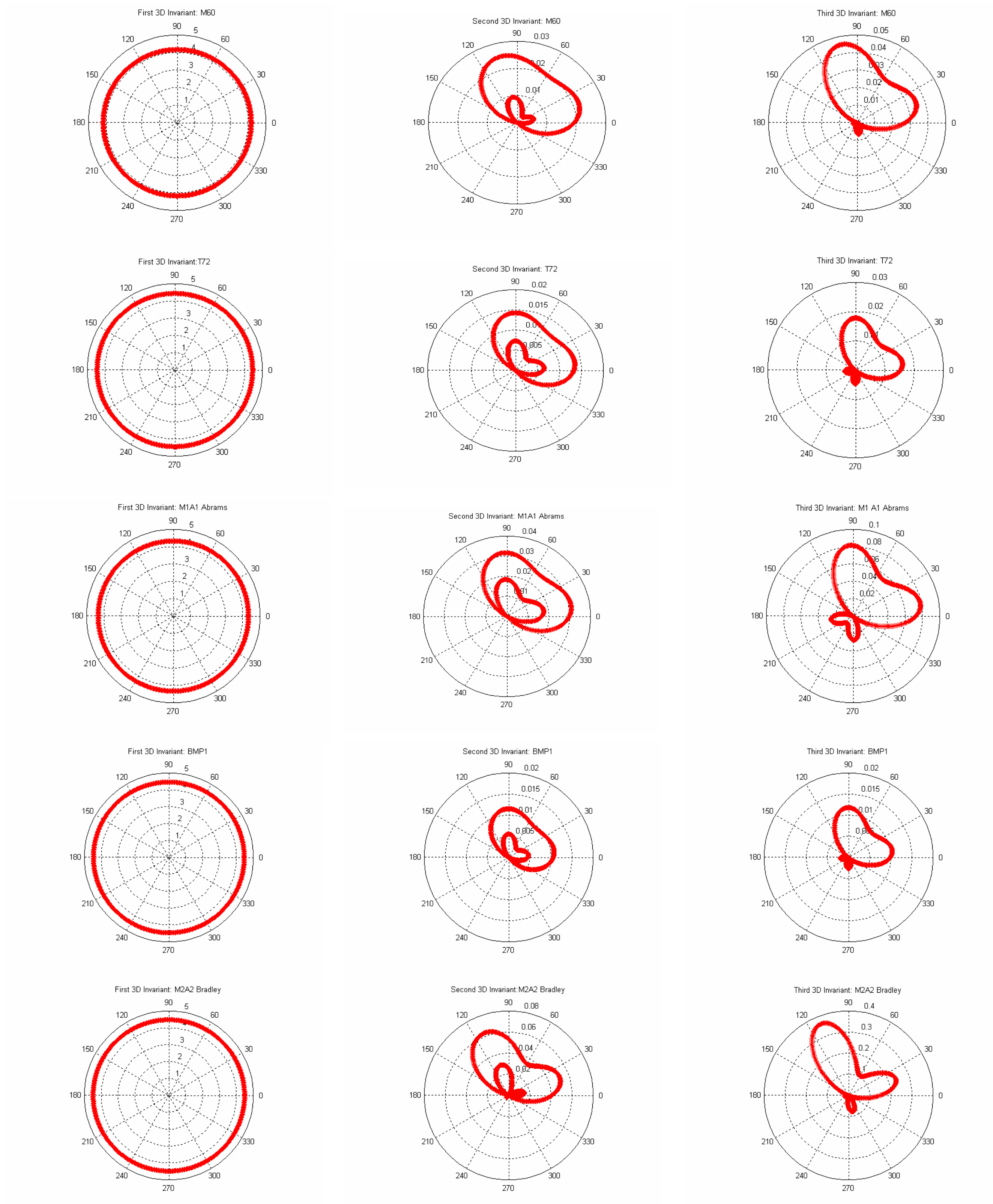
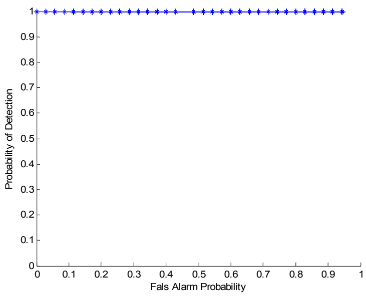
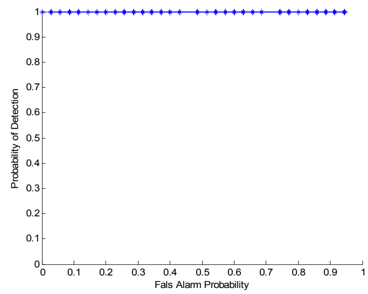


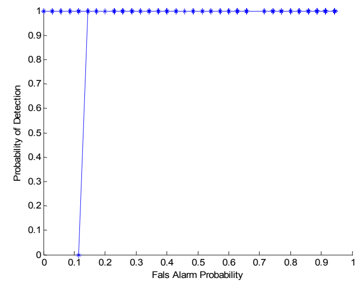
Fig.3. Plot of invariant values for 5 different targets as the targets rotate around z-axis from zero to 360 degrees



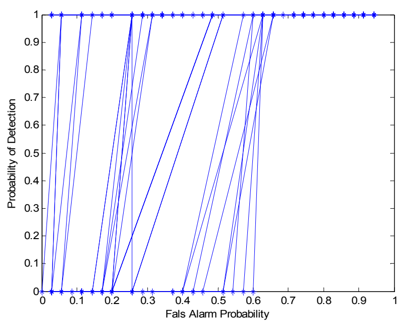
Noise Variance=0.01



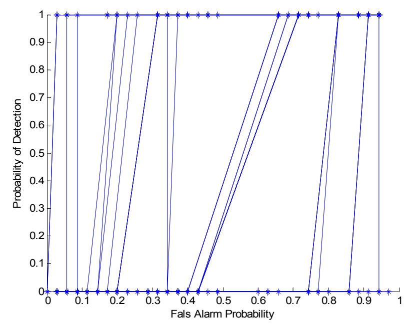
Noise Variance=0.04



Noise Variance=0.09



Noise Variance=0.25



Noise Variance=1

Fig.5. ROC Curves for Different Noise Variances for 1st Set of Invariants

Fig 4 shows how the targets are distributed in terms of their absolute invariants. The ranges of variations for different invariants seem to be different from each others.

To test the effects of coordinate transformations on the 3D invariants, for 5 typical targets, M60, T72, M1A1 Abrams, BMP1 and M2A2 Bradley, the Ladar cloud points were rotated around z-axis, from 0 to 360 degrees, by increments of 0.1 degree and at each state the set of 3D invariants were computed. The results are shown in Figure 3. The right column plots show the results for the 1st 3D invariants. As can be seen for all the five targets the 1st invariant remains very much unchanged. The second column in Fig 3 showing the plots for the 2nd 3D invariants, indicate that there some small disturbances for all of the targets. The 3rd 3D invariants, shown in the 3rd column of the Figure 3 display variations that even though small seem to be larger than those associated with the 2nd invariants. We attribute these disturbances to the quantization errors that are produced in the data due to the rotation of the coordinate system. The variations in 2nd and 3rd 3D invariants each show unique patterns that require further investigation.

By representing each target in terms of their 1st set of absolute invariant representation and using Frobenius distance as a similarity measure, we computed the probabilities of correct detection and false alarms for each of the 35 targets at various noise variances. Figure 5 shows the receiver operating characteristic curves (ROC) for all 35 targets superimposed on top of each other, at various noise variances. For noise variances below or equal to 0.09 good results are obtained. However, as noise variance is increased to 0.25 and 1 (equivalent to 1 meter error); for a good number of targets the classification performances degrade. Similarly results can be obtained for the 2nd set of absolute invariants.

5. SUMMARY

In this paper we explored the theory of invariant algebra and its applications for automatic classification of objects through their multi-dimensional signatures. The applications of binary quantics include derivation *2D invariant shape attributes*, and joint *geometrical and material invariant features.* The use of invariants of ternary quantics led to the derivation of a set of 3D invariant shape descriptors that are then used for classicization of Ladar signatures of a large class of land vehicles.

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