Schmidt orthogonal optimization of measurement matrix

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ABSTRACT

Aiming at the disadvantage of poor stability of universal measurement matrix, this paper proposes an optimization method of measurement matrix, which is called Schmidt orthogonalization method. Row vectors are orthogonalized step by step to reduce the correlation between column vectors of measurement matrix. The simulation results show that under the same sampling rate, the one-dimensional signal reconstructed based on the optimized measurement matrix has higher reconstruction accuracy and quality, and the number of measurements required for accurate reconstruction is also reduced; the two-dimensional image reconstructed based on the optimized measurement matrix has higher signal-to-noise ratio.

Keywords: Compressed sensing, signal reconstruction, measurement matrix

1. INTRODUCTION

Compressed Sensing Theory[1-3] proposes a new signal sampling theory, which makes full use of the sparseness of the signal to achieve simultaneous signal acquisition and compression, which breaks through the Nyquist sampling theorem limits. CS technology has been widely used in medical imaging, remote sensing images, high-definition film and television and other fields. The measurement matrix is one of the important contents of the compressed sensing research. In order to accurately reconstruct the signal and achieve a better compressed sensing effect, many scholars have proposed a variety of measurement matrix optimization methods[4-10].

Aiming at the shortcoming of poor stability of the universal measurement matrix, this paper proposes an orthogonalization optimization method, taking the row vector of the measurement matrix as the unit, alternately performing row vector Schmidt orthogonalization and column vector normalization. Experimental results show that in the optimized measurement matrix, the correlation between column vectors is significantly reduced, and the signal reconstruction quality is improved to varying degrees.

2. COMPRESSED SENSING FRAMEWORK

2.1 Compressed sensing theory

Assuming that a vector x is a signal, we have $x \in R^{n\times 1}$, if under a given set of bases, x can be represented by one coefficient matrix, namely: $x = \Psi\alpha$, where Ψ is the sparse transformation matrix, α is the coefficient vector, and α at most has k non-zero elements. The vector x is called the k-sparse signal.

If the measurement matrix is Φ, which not related to the sparse transformation matrix Ψ, the signal can be compressed and sampled, we can get: $y = \Phi\Psi\alpha$, where $y \in R^{m\times 1}$ is the measured value obtained by sampling. Let $\mathrm{A} = \Phi\Psi$, denoting the product of the measurement matrix and the sparse matrix, $\mathrm{A} \in R^{m\times n}$ which is called the sensing matrix. The main work of compressed sensing is to obtain the coefficient vector according to the sparse prior conditions of the signal:

$$\min \|\alpha\|_0 \quad s.t. \quad y = \mathrm{A}\alpha \tag{1}$$

The norm-based minimum optimization problem is NP-hard and cannot be solved in polynomial time. Greedy pursuit algorithms can solve such problems by selecting local optimal solutions and gradually approaching the original signal. The representative algorithm is orthogonal matching pursuit algorithm. Converting the L0 minimum norm into the L1 minimum norm problem can also be solved by the convex optimization method and transformed into a linear programming problem,

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and the representative algorithm is the basis pursuit method.

2.2 Constraint equidistance and correlation

Candes et al. proved that if the sensing matrix satisfies the Restricted Isometry Property (RIP), there is a constant $\delta_k \in (0,1)$ for any sparse signal v, and equation (2) is established, the signal can be accurately reconstructed.

$$(1-\delta_k)\|v\|_2^2 \le \|\mathbf{A}v\|_2^2 \le (1+\delta_k)\|v\|_2^2 \tag{2}$$

In practical applications, it is difficult to verify whether a certain matrix satisfies the RIP property A. In order to analyze the performance of the sensor matrix, Reference[11] proposed the concept of cross-correlation coefficient, which is used to represent the worst similarity between the row vector of the measurement matrix Φ and the column vector of the sparse transformation matrix Ψ.

Definition 1 (coherence): The coherence of the matrix, also known as the cross-correlation coefficient, is the maximum value of the inner product of any two columns of the matrix, as shown in equation (3).

$$\mu(\mathrm{A}) = \max_{1\le i,j\le m, i\ne j} \frac{|a_i^T a_j|}{\|a_i\|_2 \cdot \|a_j\|_2} \tag{3}$$

3. MEASUREMENT MATRIX OPTIMIZATION METHOD

In Euclidean space, Schmidt's orthogonalization method is suitable for linearly independent groups. A linearly independent vector group, usually the atoms are not pairwise orthogonal. When the matching pursuit algorithm is used to calculate, the error will gradually accumulate with the calculation process, which makes the calculation result unusable. The advantage of the pairwise orthogonal vector group is that the numerical calculation can maintain the stability of the result.

3.1 Schmidt orthogonalization

Schmidt orthogonalization provides a method, which is suitable for Euclidean space. Based on a set of bases of Euclidean subspace, a set of orthogonal bases of subspace can be obtained.

First, from a linearly independent vector group $\alpha_1,\alpha_2,...,\alpha_m$, we can construct an orthogonal vector group $\beta_1,\beta_2,...,\beta_m$; and secondly, we normalize the orthogonal vector group $\beta_1,\beta_2,...,\beta_m$, then we can obtain an orthonormal basis $\eta_1,\eta_2,...,\eta_m$. The mapping matrix between the orthonormal basis $\eta_1,\eta_2,...,\eta_m$ and the vector group $\alpha_1,\alpha_2,\cdots,\alpha_m$ is an upper triangular matrix.

The optimization method proposed in this paper is: firstly, we select the first row vector, then we perform norm normalization operation, after that we replace the first row of the measurement matrix with the normalized vector. Subsequently all column vectors have been normalized; secondly, we select the second row vector, then we project it onto the subspace formed by the first row vector, and then we normalize the second row vector, after that we replace the second row of the measurement matrix with the second normalized vector, and then all columns have been normalized; finally, we perform the above loop operation on all subsequent row vectors.

$$\mathbf{T} = \begin{bmatrix} \frac{1}{|\beta_1|} & -\frac{1}{|\beta_2|}\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)} & \cdots & -\frac{1}{|\beta_n|}\frac{(\alpha_n,\beta_1)}{(\beta_1,\beta_1)} \\ 0 & \frac{1}{|\beta_2|} & \cdots & -\frac{1}{|\beta_n|}\frac{(\alpha_n,\beta_2)}{(\beta_2,\beta_2)} \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & \frac{1}{|\beta_n|} \end{bmatrix}$$

3.2 Orthogonalization of the measurement matrix

References[4-11] show that doing row orthogonalization processing on the measurement matrix can effectively improve the column correlation of the Gaussian matrix while reducing the correlation between the row vectors. In general, the number of rows of the sensing matrix is less than the number of columns, so its row vectors can be regarded as a linearly independent group, which is suitable for Schmidt orthogonalization method.

Schmidt Orthogonal Optimization Algorithm
Input: Gaussian random measurement matrix. Output: the optimized measurement matrix. Step: Initialization: The initial value of the number of iterations I is set to 0, with a maximum number of iterations is 30 by default. Iteration: The following steps are performed Step 1: The Jarque-Bera test is used to calculate the number of rows and columns that each column and each row obey the Gaussian distribution; Step 2: The correlation coefficient between the column vectors is calculated and the maximum absolute value is saved; and then the correlation coefficient between the row vectors is calculated and the maximum absolute value is saved; Step 3: The modulus of each row vector is computed, and the maximum value and the minimum value are recorded; Step 4: The proposed method is applied, the row vector is normalized, and then the column vector is normalized; Step 5: The loop is exited if the condition is met, otherwise we return to Step 1.

After the experiment, the inner product between row vectors and the inner product between column vectors are used to measure the results of iterative optimization. The experimental results show that the Schmidt orthogonal optimization method proposed in this paper reduces the inner product value and achieves the purpose of reducing the coherence of the sensing matrix.

3.3 Numerical analysis after optimization of the measurement matrix

Each element in the Gaussian random matrix is a sample value from the population, so the modulus of the row vector and the modulus of the column vector of the Gaussian matrix obey the standard normal distribution after the square of the modulus of the column vector.

Therefore, the mathematical expectation of each row module of the Gaussian matrix is $\sqrt{N/M}$, and the mathematical expectation of each column module is 1.

The scale of the Gaussian matrix used in this paper is 64×256, so the mathematical expectation of each row modulus of the Gaussian matrix is 2. In the iterative operation, the square extremum of each row modulus converges to 4, which is the same as the mathematical expectation in the theoretical analysis, as shown in Figure 1.

The Jarque-Bera algorithm is used to verify that the row vector and column vector satisfy the normal distribution. As shown in Figure 2, during the orthogonalization iteration process, all the column vectors obey the standard normal distribution, and the values are not changed due to the orthogonalization optimization method. Distribution, the row vector gradually converges stably with the increase of the number of iterations, and the number of rows satisfying the normal distribution is above 98%.

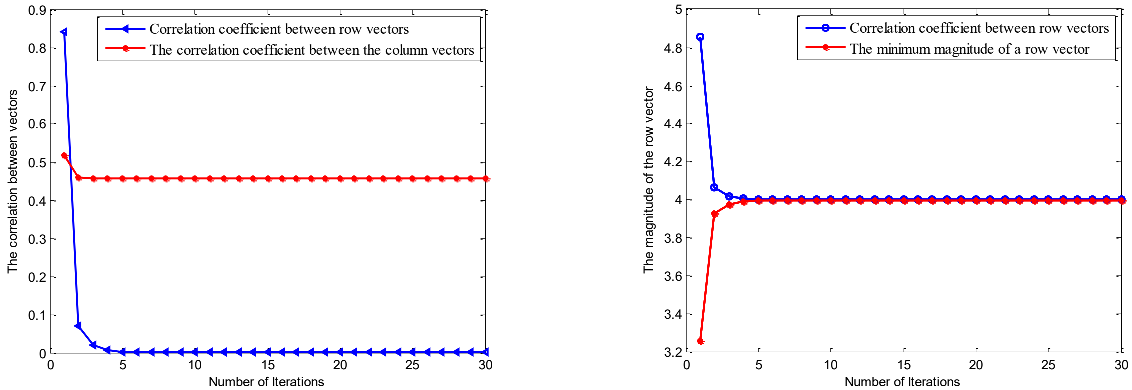


Figure 1. The relationship between correlation coefficient and modulus and the number of iterations.

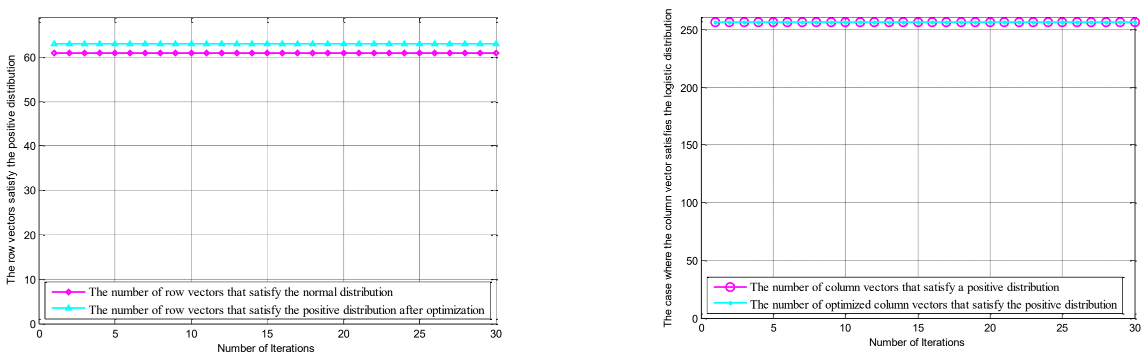


Figure 2. Row vector and column vector obey normal distribution statistics.

4. EXPERIMENT AND ANALYSIS

4.1 Single reconstruction of one-dimensional signal

The length of the original signal is 256, the sparsity is 25, and the size and location of non-zero elements obey the standard normal distribution. The sparse transformation is a standard orthogonal basis. Therefore, $A^{CS} = \Phi\Psi = \Phi I = \Phi$ in the sensor matrix and compressed sensing theory, the accurate reconstruction of the signal generally requires the number of measurements to be 3-5 times the sparseness. The number of measurements in this experiment $M = 128$ meets the conditions $M > 5 * k$ for accurate reconstruction of the OMP algorithm. The measurement matrix adopts the commonly used Gaussian random matrix with a size of $128 * 256$.

Figure 3 shows the effect of a single signal reconstruction. It can be found that both the original Gaussian measurement matrix and the optimized measurement matrix can reconstruct the original signal. Using the optimized measurement matrix, the error between the reconstructed signal and the original signal is smaller than the error computed with original.

Since the sparse signal and the measurement matrix are generated randomly, the results of a single experiment are not convincing enough. In order to test the performance of the matrix, the OMP algorithm is used to repeat the experiment 100 times to calculate the average reconstruction error. The average error of the Gaussian random matrix is 12.91, the average error of the optimized matrix is 7.23, and the matrix optimization can indeed improve the signal reconstruction performance.

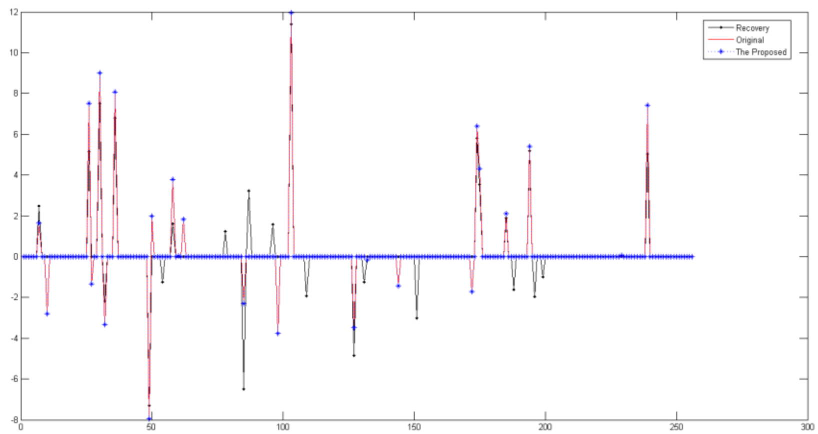


Figure 3. One-dimensional sparse signal reconstruction comparison.

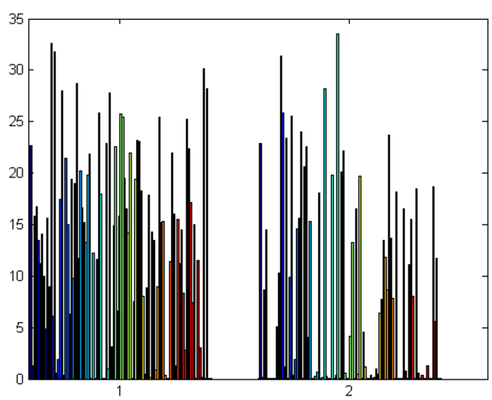


Figure 4. Numerical comparison of 100 reconstruction errors of one-dimensional signal.

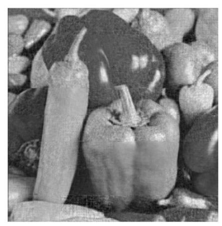
Figure 4 is a histogram of 100 reconstruction errors. The left side is the reconstruction error computed by traditional Gaussian measurement matrix, and the right is reconstruction error computed by the optimized measurement matrix. It can also be seen intuitively that the error value computed by the optimized measurement matrix is generally lower than the traditional Gaussian measurement matrix.

4.2 Image reconstruction

Two-dimensional images are used for simulation. First, 256*256 images Peppers is sparsely sampled. The image is divided into blocks of non-overlapping sizes, and each block is arranged into a column vector with a length of 1024. The sparse base adopts the DCT base and the sparsity is 200.

The measurement matrices are respectively Gaussian random matrix, Bernoulli matrix, and Toplitz matrix for testing. The size of the Gaussian random matrix is $M \times N, M = \text{compresion\_rate} \times N, N = 256$, the mean is 0, and the variance is $1/M$; the size of the Bernoulli matrix is $M \times N$, and the variance is $\pm 1/\sqrt{M}$; the Toplitz matrix first constructs a free vector u, and a square matrix $N \times N$ is generated from the vector u, randomly extract M rows to form a measurement matrix. The objective evaluation standard peak signal to noise ratio (peak signal to noise ratio, PSNR) is used to evaluate the quality of image restoration.

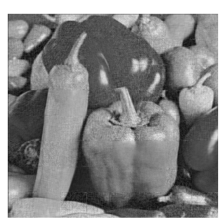
(a) Gaussian random matrix PSNR31.3873



(c) Random Bernoulli matrix PSNR32.0541

(e) Toplitz matrix PSNR29.4439

(b) Gaussian matrix after orthogonal optimization PSNR32.0049

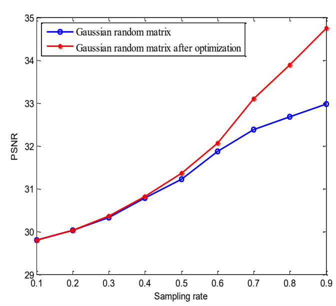


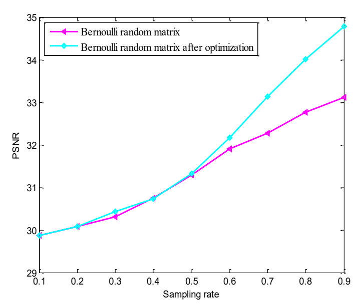
(d) Bernoulli matrix after orthogonal optimization PSNR32.8638

(f) Toplitz matrix after orthogonal optimization PSNR29.9385

Figure 5. Image reconstruction.

Using the optimized measurement matrix for image reconstruction, the PSNR of the image has been improved to varying degrees. It can be seen from Figure 5 that when the sampling rate is less than 0.4, the PSNR of the image is basically unchanged; when the sampling rate is 0.6, for the Peppers image, after the Gaussian random matrix is optimized, the image quality is improved by 0.66dB, the Bernoulli matrix is improved by 0.76dB, and the Toplitz matrix is improved by 0 .83dB.





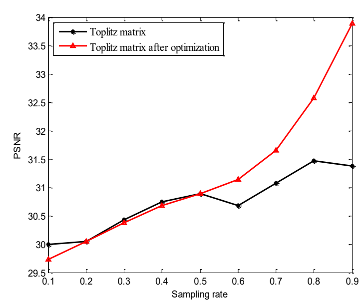


Figure 6. Peppers Image reconstruction PSNR.

Table 1. Peppers image reconstruction (PSNR).

Matrix type	**Compression ratio**				
	0.1	0.3	0.5	0.7	0.9
Gaussian	29.8054	30.3228	31.2219	32.37	32.9701
Gaussian_opt	29.8019	30.3545	31.3659	33.0905	34.7510
Bernoulli	29.8605	30.3088	31.2901	32.2794	33.1122
Bernoulli_opt	29.8730	30.4321	31.3320	33.1285	34.785
Toplitz	29.9954	30.4264	30.8877	31.0698	31.3794
Toplitz_opt	29.7352	30.3801	30.8915	31.655	33.8824

Figure 6 shows the numerical change of the PSNR of the Peppers image. The detailed data is shown in Table 1. It can be seen that as the sampling rate increases, the image reconstruction quality improves more.

5. CONCLUSION

In compressed sensing theory, the design and optimization of the measurement matrix are the key factors for signal reconstruction. The performance of the measurement matrix depends on the column coherence of the measurement matrix. The Schmidt orthogonalization optimization algorithm proposed in this paper reduces the column coherence of the random matrix, and the algorithm has strong universality. It is applicable to Gaussian random matrix, Bernoulli matrix, Toplitz matrix, etc., one-dimensional signal reconstruction and two-dimensional signal reconstruction. Numerical experiments on two-dimensional images have proved the effectiveness of the method, and the accuracy of signal reconstruction has been improved.

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